Arbitrary Lagrangian-Eulerian Schemes for Ocean Modelling & A Few Memories of Unstructured Mesh Methods for CFD

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I.

1. CFD – (Not) Geophysical Fluid Dynamics

'Conventional' CFD differs from GFD in a number of important ways:

Pressure Coupling

Velocity-pressure coupling is 'isotropic' – no hydrostatic assumption:

- Non-hydrostatic pressure distribution computed at each time-step.

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Resolve boundary-layer flows through mesh adaptation:

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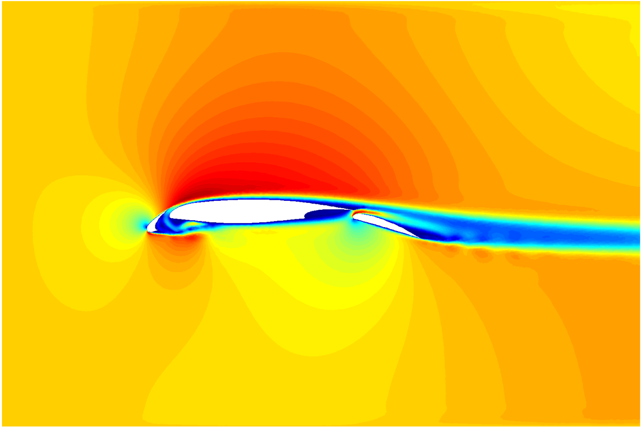
Geometric Constraints

Solve flow problems for arbitrarily complex geometries.

- Use unstructured meshes and numerical methods.

1. CFD – Ubiquitous Aerodynamics Studies



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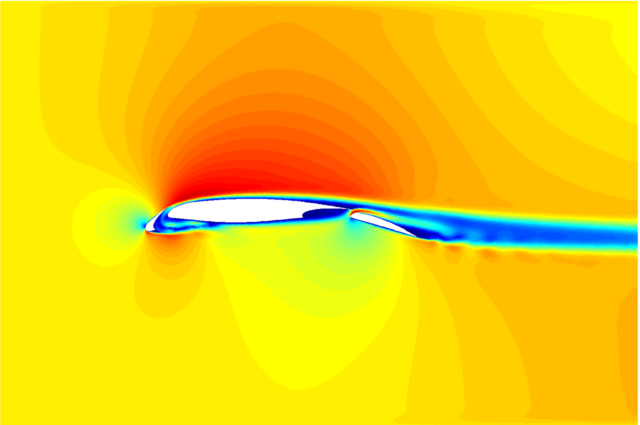

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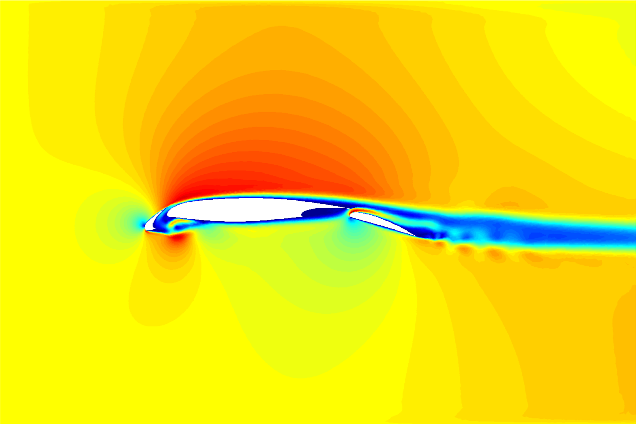

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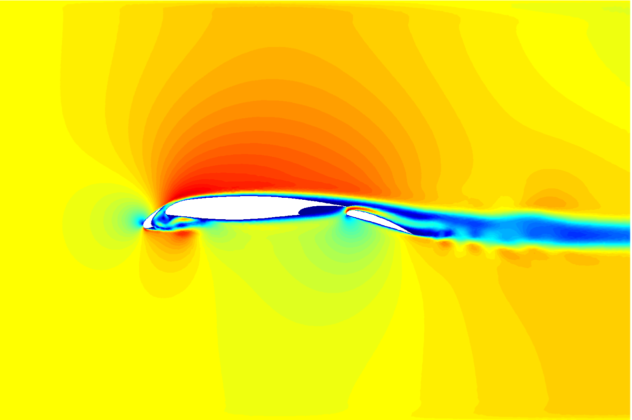

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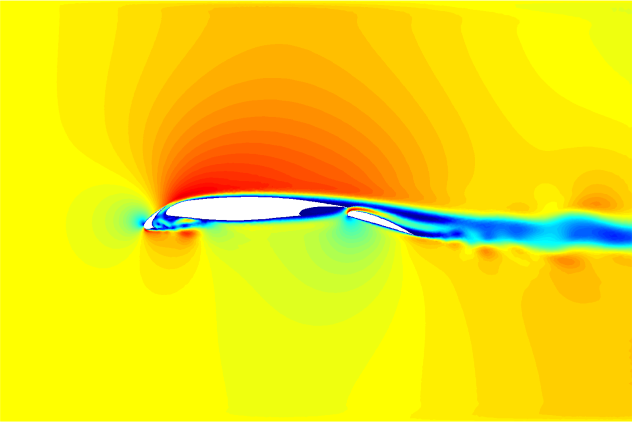

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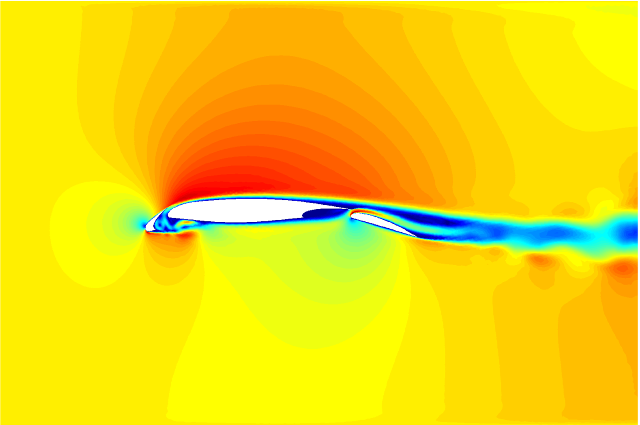

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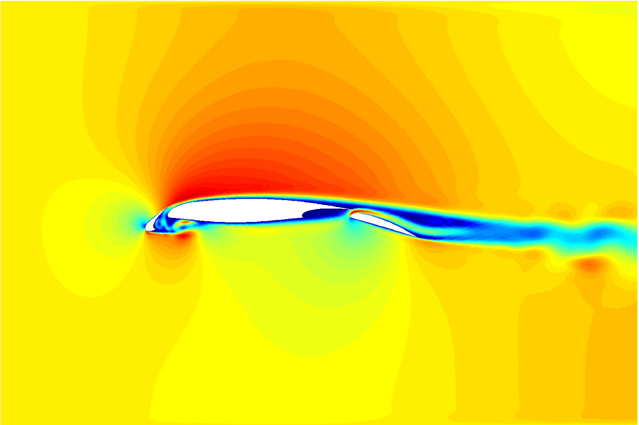

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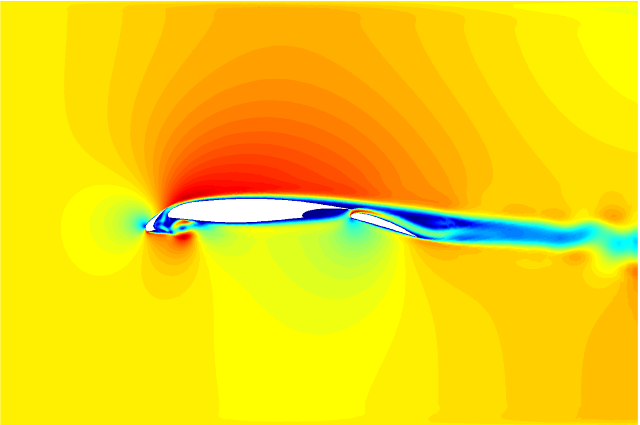

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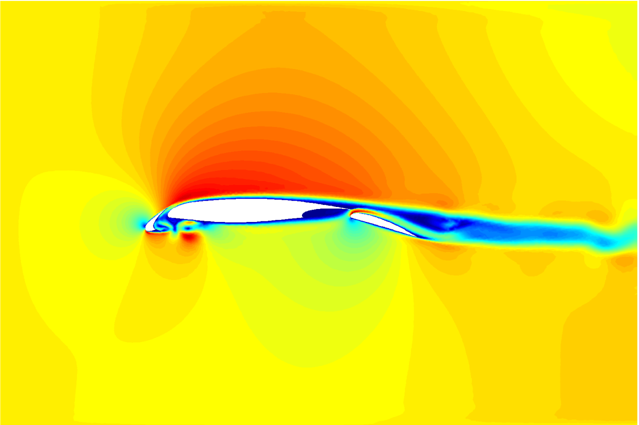

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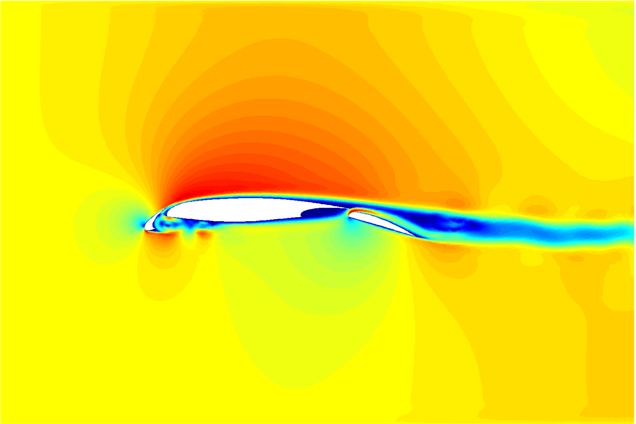

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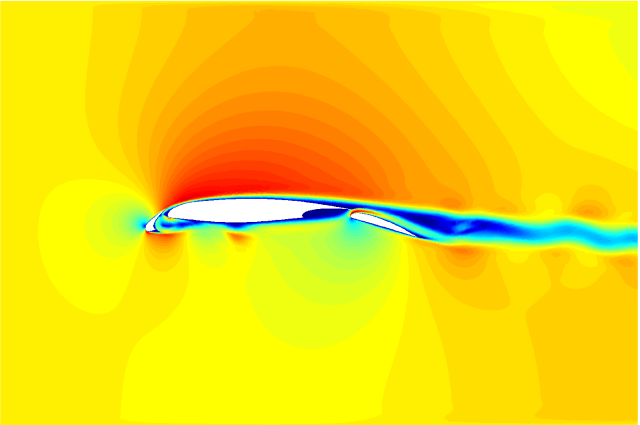

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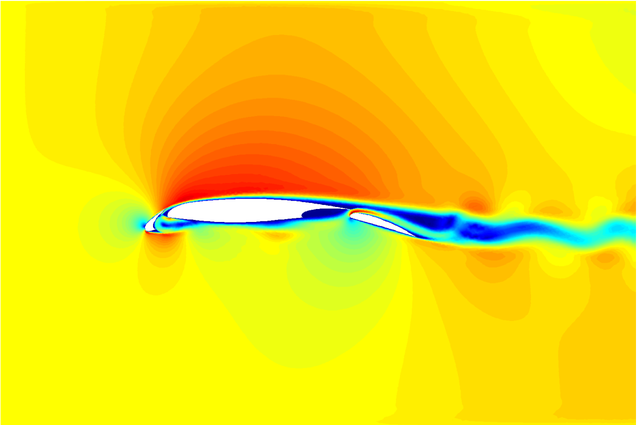

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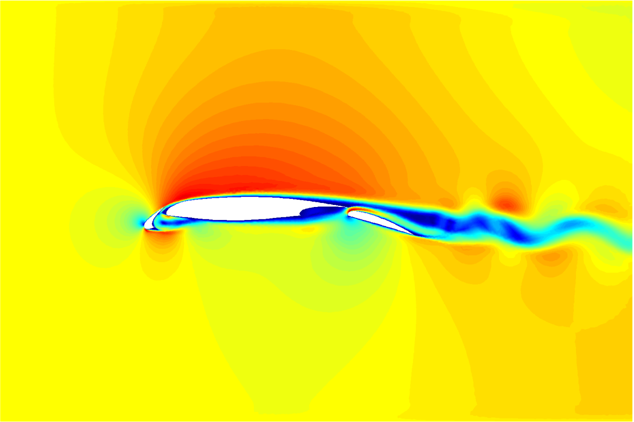

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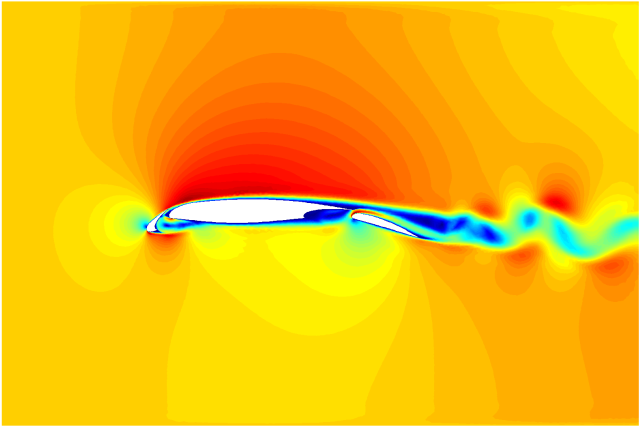

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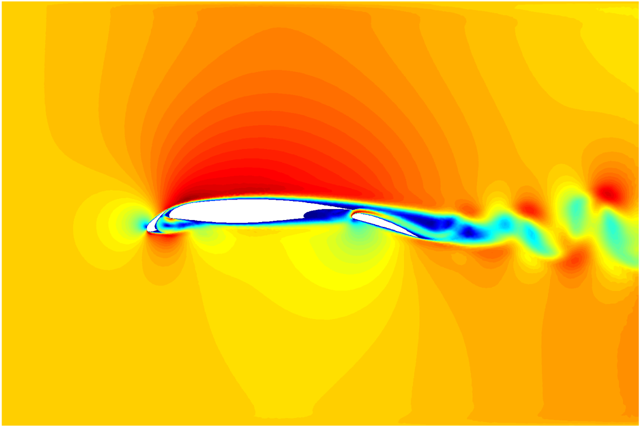
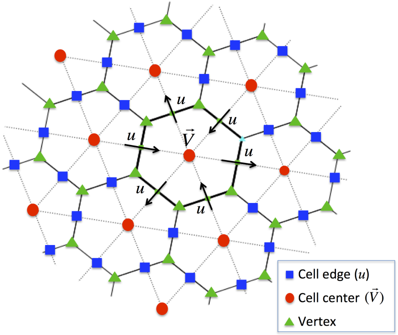


1. CFD – Voronoi-based Finite Volume Schemes

Integrate equations of motion in divergence form over control volumes:

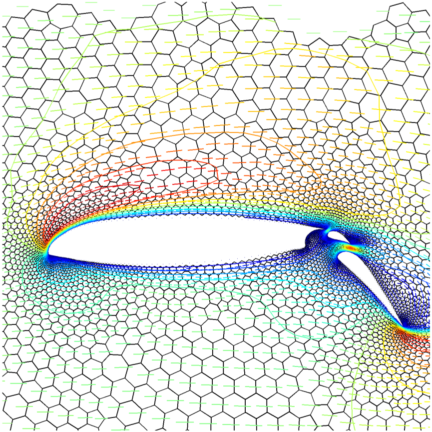
$$\int_\Omega \frac{dq}{dt} + \nabla \cdot (\mathbf{F}(q)) - \mathbf{S_q}\, dV = 0$$

- A Voronoi diagram is a set of polygonal cells.
- Each cell contains varying numbers of edges.
- The edges of each cell are always **orthogonal** to a common centre.
- The Voronoi diagram is constructed upon an underlying triangulation.



1. CFD – Voronoi Finite-Volumes

Variable resolution Voronoi mesh, clustering elements in boundary layer regions.



2. Mesh – Unstructured Triangulations

The creation of 'optimal' unstructured triangulations & Voronoi diagrams is non-trivial:

- Need to ensure that 'element-quality' is adequate:
  - Don't want highly skewed cells – aim for equilateral triangles.
  - Don't want cell size to vary too rapidly.
- Need to optimise both vertex positions and mesh topology.

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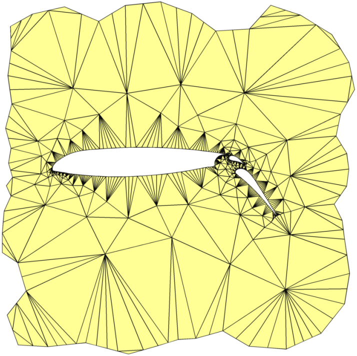
- Need to optimise both vertex positions and mesh topology.

The so-called **Delaunay Triangulation** offers a convenient framework for mesh generation. Given a set of vertices $X \subset \mathbb{R}^d$, the Delaunay triangulation $\mathcal{T} = \mathrm{Del}(X)$ is known to be 'optimal' for a range of geometric criteria.

2. Mesh – Quality Delaunay Triangulations

'Refinement' algorithms incrementally add vertices to a coarse mesh until all constraints are satisfied:

- A coarse triangulation is built based on the external geometry of the domain.



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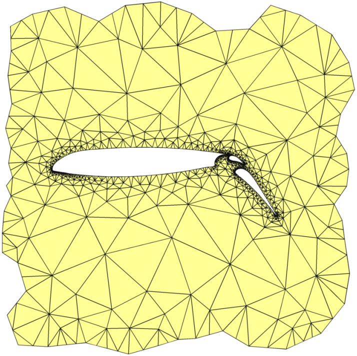

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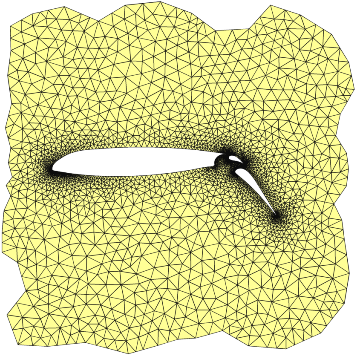
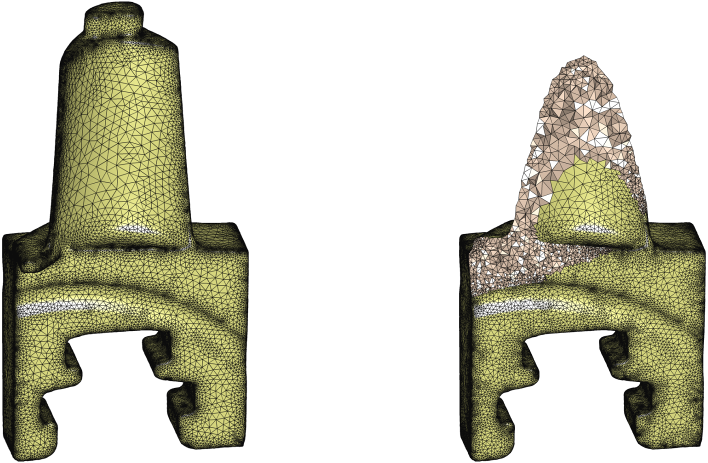
- A coarse triangulation is built based on the external geometry of the domain.
- Additional vertices are added to 'remove' any poor quality triangles by splitting them.
- All elements in the final mesh satisfy shape and size constraints. In $\mathbb{R}^2$, the refinement algorithm can achieve a minimum angle $\theta_{\text{min}} \geq 30^\circ$.

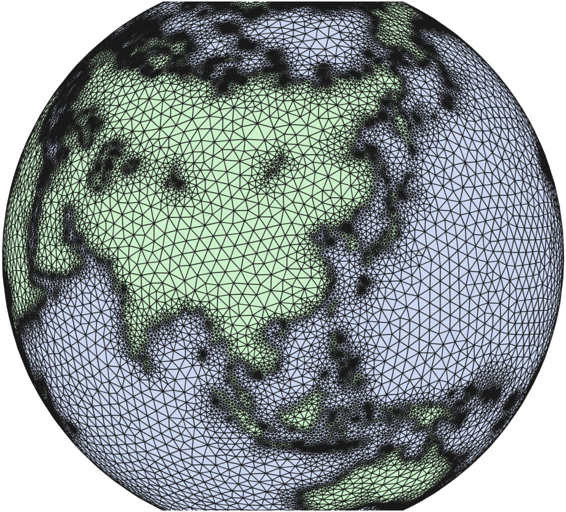


2. Mesh – Surface & Volume Triangulations

Surface and volumetric triangulations of a turbine blade for a 3d CFD study.



Unstructured GFD Applications?



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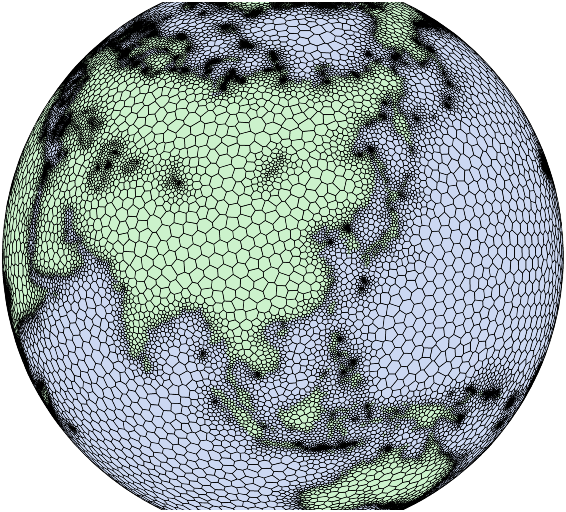

II.

3. ALE – Eulerian vs Lagrangian Methods

Equations of motion can be represented in either an Eulerian or Lagrangian form:

- **Eulerian Form**: Mesh is fixed and transport is achieved through explicit evaluation of cell-wise fluxes.
- **Lagrangian Form**: Mesh moves with the flow. Flux evaluation is replaced by mesh movement.

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Lagrangian methods allow the mesh to align locally with features of the flow:

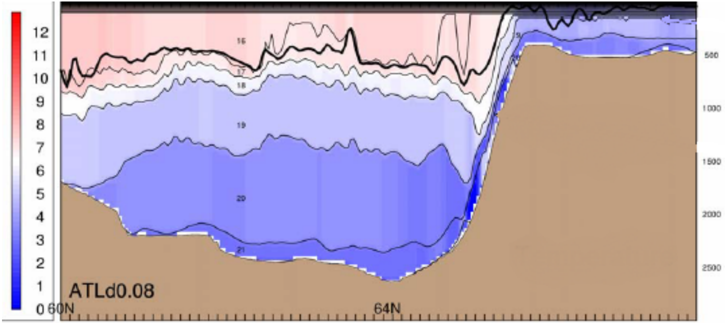
Quasi-Isopycnal Representation

Lagrangian vertical transport can be used to achieve a quasi-isopycnal representation in the open-ocean, where the flow is essentially adiabatic. **Minimisation of spurious vertical mixing.**

3. ALE – Layered Vertical Structure

The aim is to follow the approach of HYCOM, introducing a 'flexible' vertical discretisation that:

- Follows isopycnals where possible.
- Smoothly transitions to other representations where necessary. (z-model in mixed layer, σ-model near sharp topography, etc).



[1]

[1] Temperature profiles from Bleck 2004

3. ALE – Layer-wise Equations of Motion

The equations of motion for the ocean can be written as a set of layer-wise conservation laws:

$$\frac{d\mathbf{u}_h}{dt} + \nabla \cdot (\mathbf{u}_h \, \mathbf{u}_h^{\mathbf{T}}) = -\nabla_p(\Phi) + \mathbf{S}_{\mathbf{u_h}}$$

$$\frac{d\Phi}{dp} = -\alpha$$

$$\frac{d\Delta p}{dt} + \nabla \cdot (\mathbf{u}_h \Delta p) = \mathbf{S}_{\mathbf{p}}$$

$$\frac{d\theta, S}{dt} + \nabla \cdot (\mathbf{u}_h \, \theta, S) = \mathbf{S}_{\theta,\mathbf{S}}$$

Rather than introducing a 'hybrid' vertical coordinate (as per HYCOM), we instead form a finite-volume scheme, integrating over layers of variable thickness.

3. ALE – Arbitrary Lagrangian Eulerian (ALE) Methods

Issues can arise with purely-Lagrangian methods due to the movement of the grid:

- The grid may become overly distorted due to local flow characteristics.
- The grid may evolve into a non-optimal configuration.

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- The grid may become overly distorted due to local flow characteristics.
- The grid may evolve into a non-optimal configuration.

These issues can be mitigated through use of an **Arbitrary Lagrangian Eulerian (ALE)** approach:

Quasi-Eulerian Re-mapping

If the grid is 'far-enough' away from optimal, **re-map** all flow variables onto a new target grid via interpolation.

3. ALE – Simple Sketch of an ALE Algorithm

$$\Phi^t = \Phi_b - \int_{p_b}^{p_t} \alpha(\theta^t, S^t, p^t)\, dp$$

$$\Delta p\, \mathbf{u}_h^{t+\delta t} = \mathbf{u}_h^t + \delta t(-\nabla_p \Phi^t - \nabla \cdot (\Delta p\, \mathbf{u}_h\, \mathbf{u}_h^{\mathbf{T}})^t + \Delta p\, \mathbf{S}_u)$$

$$\Delta p^{t+\delta t} = \Delta p^t - \delta t \nabla \cdot (\Delta p\, \mathbf{u}_h^{t+\delta t})$$

$$(\Delta p\, \theta, \Delta p\, S)^{t+\delta t} = (\Delta p\, \theta, \Delta p\, S)^t - \delta t \nabla \cdot (\,\Delta p\, \mathbf{u}_h^{t+\delta t}\, (\theta^t, S^t))$$

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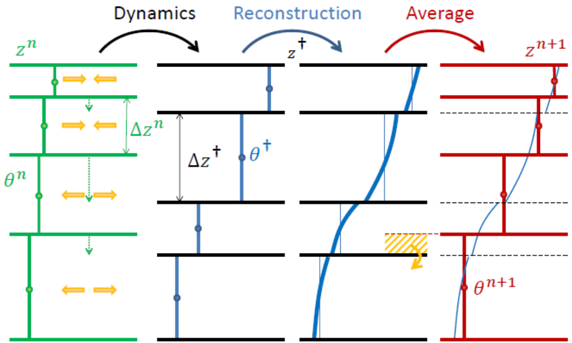
At $t + \delta t$ the grid has drifted (due to vertical transport):

If the grid is not where we want it, we can **re-map** all flow variables onto a new grid via a (conservative) **interpolation scheme**.

3. ALE – Column-wise Sketch of an ALE Algorithm

- **Update** flow variables and cell thickness via Lagrangian motion.
- **Reconstruct** cell-wise polynomials on current mesh.
- **Integrate** polynomials over new mesh to get new cell means.



[a]

[a]from Adcroft 2013

3. ALE – Evaluating the Pressure Gradient Force

Considering that the Δp layers are **sloping** and **non-uniform** in thickness, evaluation of the pressure gradient term $\nabla_p(\Phi)$ is non-trivial.

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A well-known approach approximates the pressure gradient on a sloping layer '**s**' **directly**, as a finite-difference of the Montgomery potential **M**:

$$M = \alpha \nabla_s(p) + \nabla_s \Phi$$

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Due to non-linearities in the equation of state $\alpha(\theta, S, p)$, such an approach is not typically stable. In regions of **sharp topography** and **stratification**:

- A small fraction of the vertical force balance can 'contaminate' the horizontal.
- Such occurances can cause spurious 'spontaneous motion' from an equilibriated state.

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Following an approach of Adcroft et al. [1], the pressure gradient can instead be evaluated **indirectly**, via a finite-volume integral:

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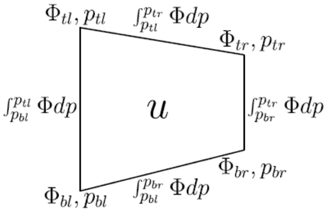
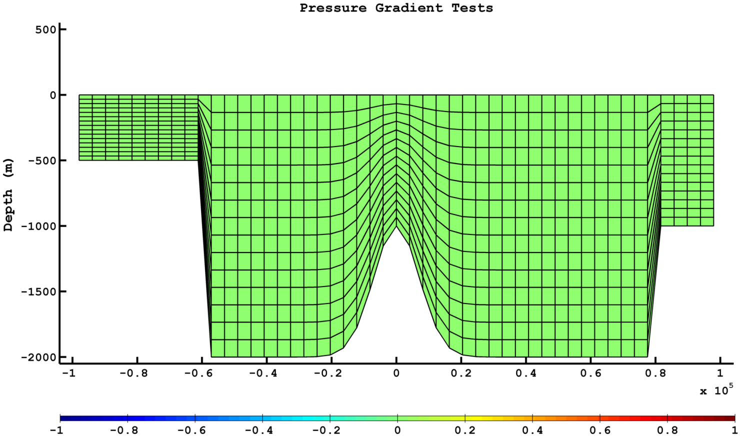
This formulation accounts for the fully non-linear distribution of Φ around each element:



3. ALE – Evaluating the Pressure Gradient Force

Assess spurious motion with variable topography, linear stratification.



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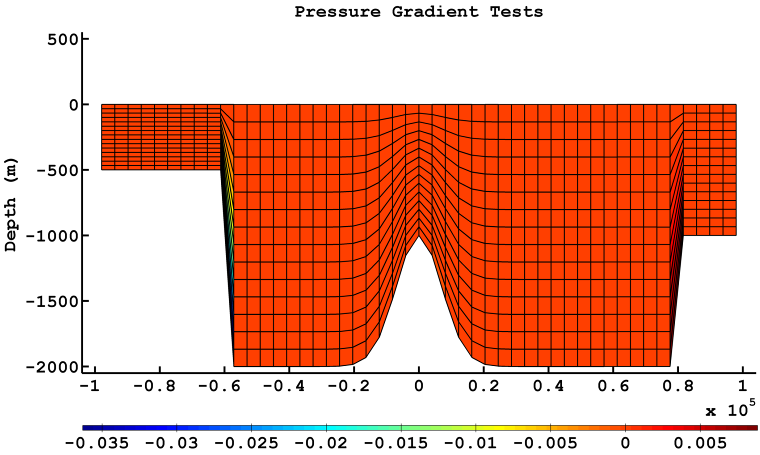

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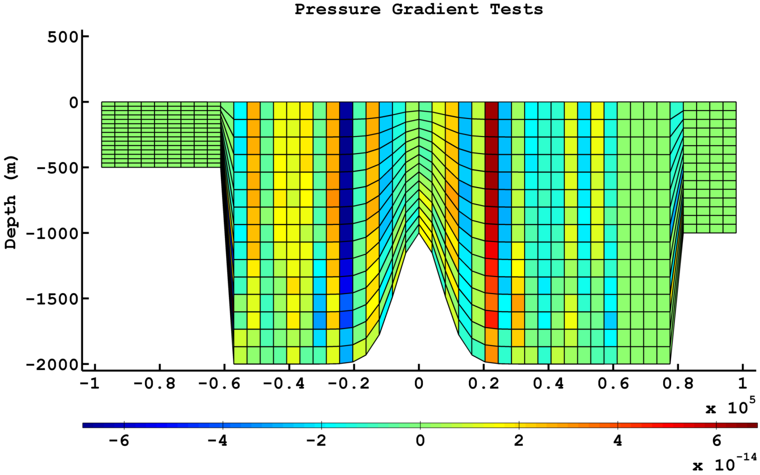


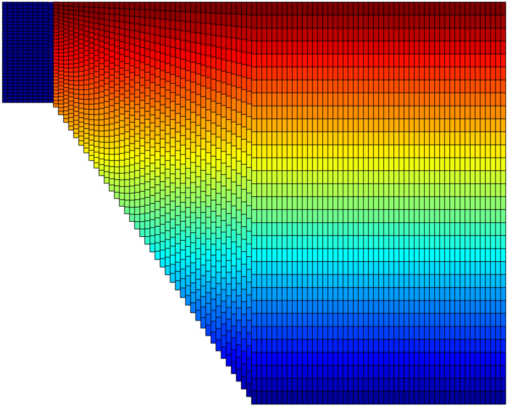
3. ALE – Evaluating the Pressure Gradient Force

Given a sufficiently high-order quadrature, the finite-volume pressure gradient formulation achieves $\approxeq 0.0$ error.

Such a scheme allows flexible vertical discretisation, but will maintain equilibrium in the presence of sharp topography ad stratification.

3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.



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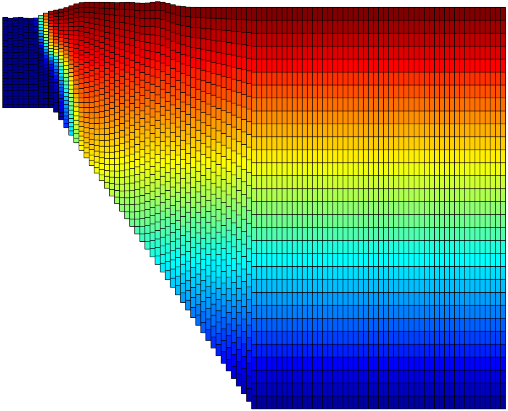

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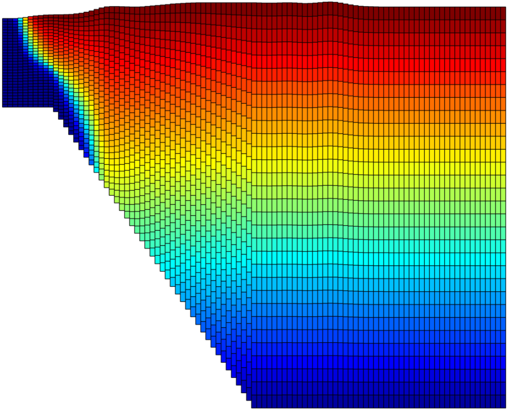

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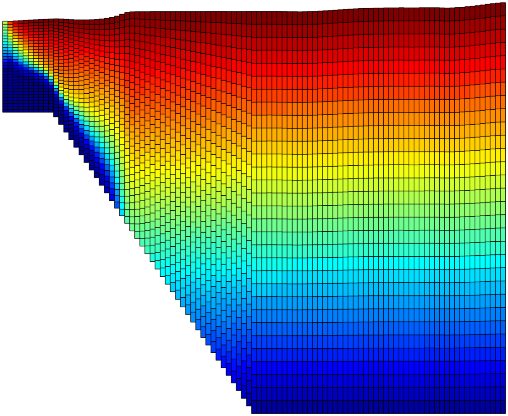

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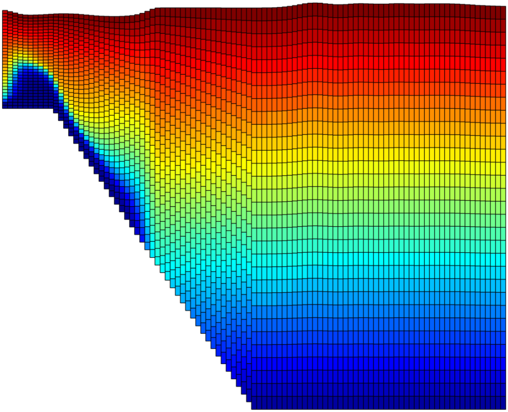

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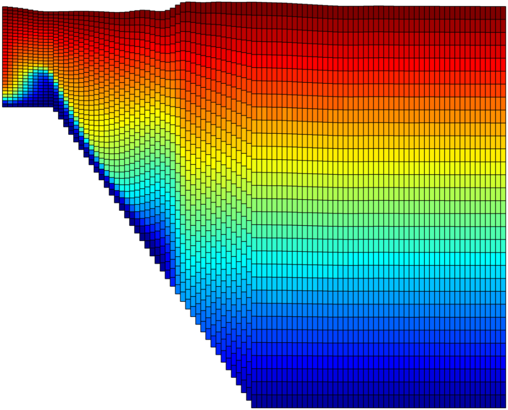

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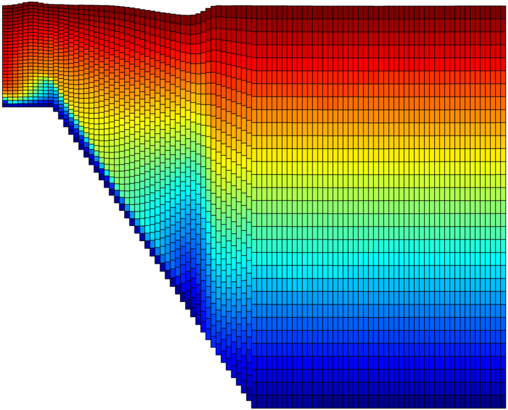

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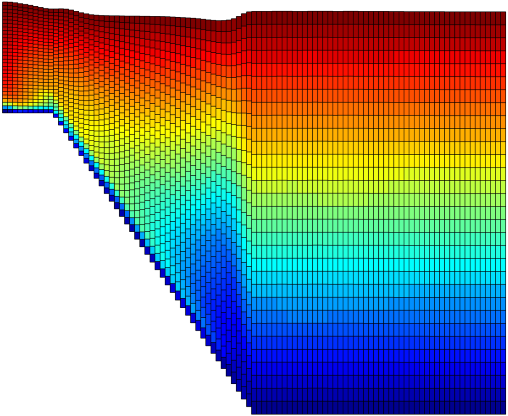

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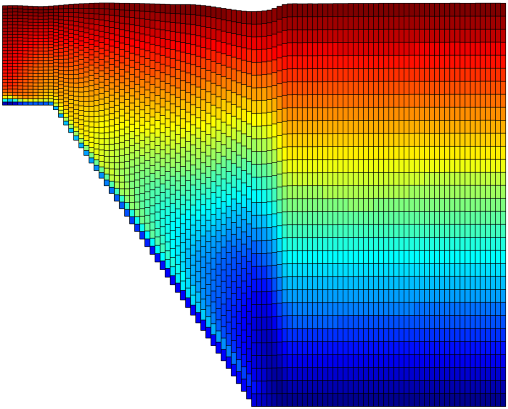

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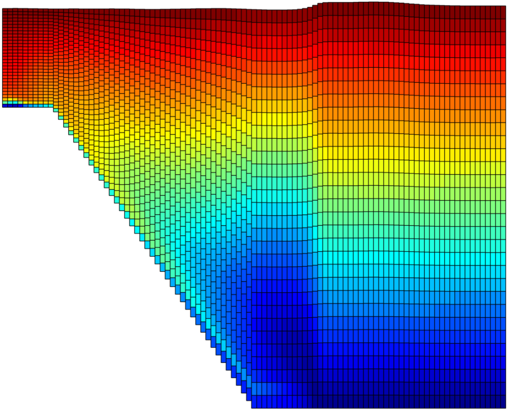

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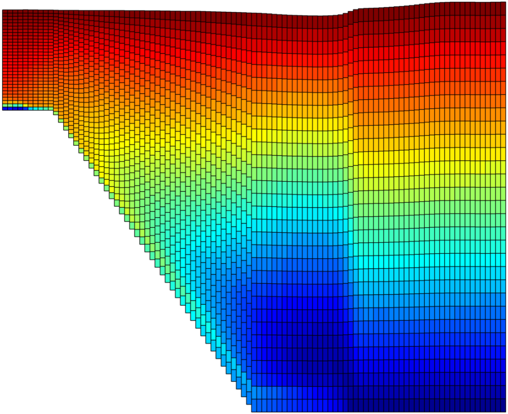

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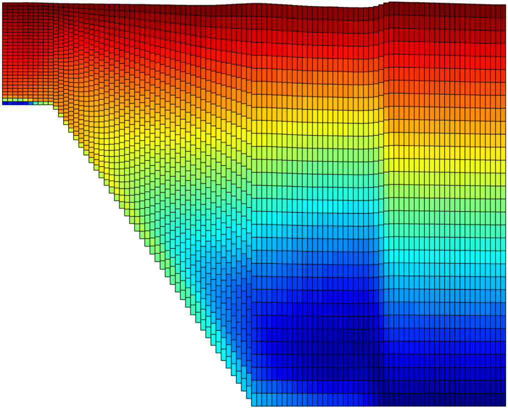

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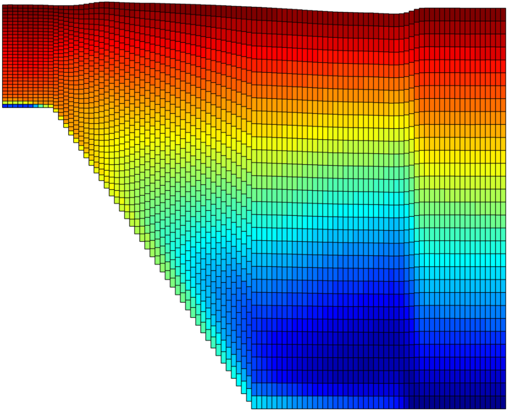

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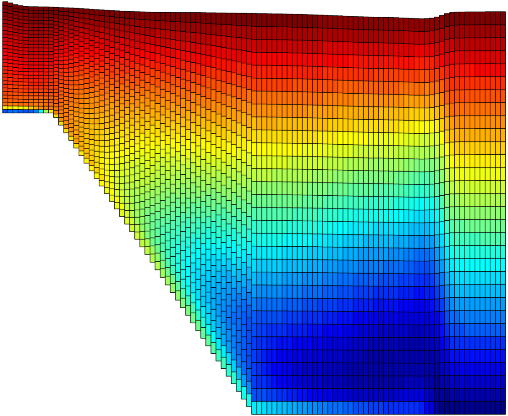

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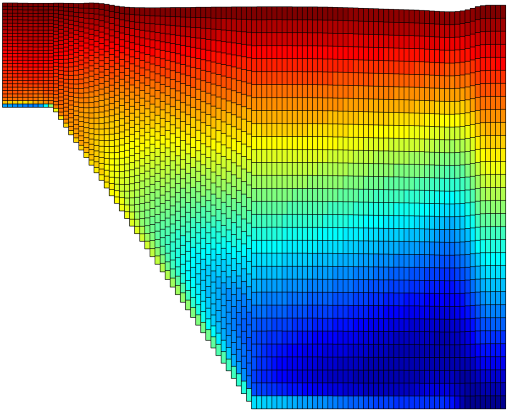

4. Summary

- Developed a simple 'proof-of-concept' layered ocean model using ALE methodologies.
- Developed a stable pressure gradient formulation that minimises pressure gradient errors with arbitrary layer geometries/stratification.
- Looking to improve 2D model:
  - Variable number of layers per column.
  - General boundary conditions.
  - Sub-grid parameterisations.
- Incorporate ALE technology into the next iteration of the GISS ocean model.

References

[1] – Adcroft, A. and Hallberg, R. and Harrison, M., A finite volume discretization of the pressure gradient force using analytic integration. Ocean Modelling, 2008.